Q. Evidence of Dr. Vander Weide: Please provide a copy of Principles for Lifetime Portfolio Selection: Lessons from Portfolio Theory. . .” noted at p. 89 of 106.

A. A copy of Dr. Vander Weide’s article is attached.

Principles for Lifetime Portfolio Selection:
Lessons from Portfolio Theory

Principles for Lifetime Portfolio Selection: Lessons from Portfolio Theory*

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Abstract

Portfolio theory is concerned with developing general principles and practical models for making sound lifetime portfolio decisions. Much of the current research on portfolio theory emanates from the path-breaking mean-variance portfolio model of Nobel Laureate Harry Markowitz. Although the mean-variance model continues to be the most widely used portfolio model in financial practice, economists have devoted considerable effort to research on two additional models of portfolio behavior, the geometric mean model and the lifetime consumption-investment model. These models are also useful to investors because they offer significant additional insights into optimal portfolio behavior. The purpose of this paper is to review the major findings of the research literature on the mean-variance model, the geometric mean model, and the lifetime consumption-investment model, and, on the basis of this review, to develop a set of practical guidelines for making lifetime portfolio decisions.

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I. Introduction

1. An individual's savings and investment choices at various stages of life are among the most important decisions he or she can make. A person entering the workforce in 2008 can expect to work for approximately 40 to 45 years and to live in retirement for an additional 20 to 25 years. During his working life, an individual must accumulate sufficient assets not only to live comfortably in both good and bad economic times, but also to live comfortably in retirement. To achieve the goal of maximizing economic welfare over his expected lifetime, an individual consumer/investor should have a sound understanding of the basic economic principles of lifetime portfolio selection.

2. The lifetime consumption/investment decision problem is complex. Suppose an individual consumer/investor divides her expected remaining lifetime into N equal periods. At the beginning of period 1, she must allocate her wealth W_0 to consumption C_0 and investment W_0-C_0. Her wealth at the beginning of the next period, W_1, will depend on both the amount she chooses to invest at the beginning of period 1 and the return she earns on her investment. The consumer/investor recognizes that she will continue to make consumption/investment decisions at the beginning of each period of her life. Her goal is to maximize her expected utility from lifetime consumption. Since expected utility from lifetime consumption depends on consumption/investment decisions in every period of her life, and the opportunities in later periods depend on the results of decisions in earlier periods, the individual consumer/investor must potentially solve a complex N period optimization problem simply to make the correct consumption/investment decision at the beginning of period 1.

3. Portfolio theory is concerned with developing general principles and practical models for making sound lifetime portfolio decisions. Much of the current research on portfolio theory emanates from the path-breaking mean-variance portfolio model of Nobel Laureate Harry

Markowitz. Markowitz (1952, 1959) recommends that in making investment decisions, investors should explicitly recognize investment risk as measured by variance of return, as well as expected return. He describes how the variance of return on a portfolio of securities depends on the amount invested in each security, the variance of return on each security, and the correlation between the returns on each pair of securities. He also suggests that investors limit their choices to an efficient set of portfolios that provide the highest mean return for any level of variance and the lowest variance of return for any level of mean. By providing an intuitively appealing measure of portfolio risk and a framework for analyzing the basic risk/return tradeoff of portfolio decisions, Markowitz revolutionized both the theory and practice of portfolio management. For that reason, Markowitz is properly called the father of Modern Portfolio Theory.[1]

4. The beauty of the Markowitz mean-variance model lies in its blend of elegance and simplicity. Markowitz achieves elegance by providing investors a sophisticated tool for: (i) understanding how portfolio mix decisions affect portfolio risk; and (ii) determining those portfolios that provide an efficient combination of risk and return. He achieves simplicity by focusing solely on the economic trade-off between portfolio risk and return in a single-period world.[2]

5. Although the mean-variance model continues to be the most widely used portfolio model in financial practice, economists have devoted considerable effort to research on two additional models of portfolio behavior, the geometric mean model and the lifetime consumption-investment model. These models offer significant additional insights into optimal

[1] This paper is dedicated to Dr. Markowitz in celebration of his 80th birthday.

[2] Markowitz discusses many of the dynamic economic forces that affect lifetime consumption and investment decisions in the later chapters of Markowitz (1959). However, the economic forces described in this discussion are not incorporated directly in his single-period mean-variance model.

portfolio behavior. The purpose of this paper is to review the major findings of the research literature on the mean-variance model, the geometric mean model, and the lifetime consumption-investment model, and, on the basis of this review, to develop a set of practical guidelines for making lifetime portfolio decisions.

II. The Markowitz Mean-variance Model

6. Investors make portfolio decisions by selecting the securities to include in the portfolio and the amount to invest in each security. In making risky portfolio choices, the Markowitz mean-variance approach assumes that investors: (1) consider only the mean and variance of the probability distribution of portfolio and security returns; (2) for a given level of mean return, prefer a portfolio with a lower variance of return; and (3) for a given level of variance of return, prefer a portfolio with a higher mean return.

7. As Markowitz demonstrates, the above assumptions suggest that an investor's portfolio decision problem can be solved in three steps. First, an investor can estimate the mean and variance of return on each security and the correlation of returns on each pair of securities. Second, an investor can calculate the mean and variance of return on each feasible portfolio and determine an "efficient frontier" of portfolios that offer the lowest variance of return for any level of mean return and the highest mean return for any level of variance of return. Third, an investor can choose a portfolio on the efficient frontier. This paper will focus primarily on steps two and three.

A. Estimating the Mean and Variance of Portfolio Returns

8. Assume that there are N securities and that an investor allocates the proportion X_i of his wealth to security i. Let R_i denote the return on security i and R_p the return on the portfolio of N securities. Then:

$$R_p = R_1 X_1 + R_2 X_2 + \cdots + R_n X_n, \quad \textbf{(1)}$$

where R_p and $R_1 \ldots R_n$ are random variables.

9. According to Equation (1), the portfolio return, R_p, is a weighted average of the returns on the securities in the portfolio. Formulas for calculating the mean and variance of a weighted sum of random variables are presented in most introductory probability texts. Using these formulas, the mean of the portfolio return, E_p, is given by:

$$E_p = E_1 X_1 + E_2 X_2 + \cdots + E_n X_n, \quad \textbf{(2)}$$

where $E_1, \ldots, E_n$, are the mean, or expected, returns on the individual securities; and the variance of the portfolio return is given by:

$$V_p = \sum_i V_i X_i^2 + \sum_i \sum_{j>i} 2 C_{ij} X_i X_j, \quad \textbf{(3)}$$

where V_i is the variance of return on security i, and C_{ij} is the covariance of returns on security *i* and security *j*.

10. In the Markowitz mean-variance model, investment risk is measured by either the variance of the portfolio return or its equivalent, the standard deviation of portfolio return.[3] The formula for portfolio variance, Equation (3), can be used to provide insight on how investors can reduce the risk of their portfolio investment. Recall that the covariance of returns on security i and security j can be written as the product of the standard deviation of return on security i, SD_i, the standard deviation of the return on security j, SD_j, and the correlation of returns on securities i and j, ρ_{ij}:

$$C_{ij} = SD_i \times SD_j \times \rho_{ij}. \quad \textbf{(4)}$$

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[3] Variance and standard deviation of return are considered to be equivalent measures of risk because the standard deviation is the positive square root of the variance, and the positive square root is an order-preserving transformation. Thus, portfolios that minimize the variance of return for any level of mean return will also minimize the standard deviation of return for any level of mean return.

To simplify the analysis, assume that: (i) the variances on all securities are equal to the average security variance, $\overline{V}$; (ii) the correlation of returns on all securities i and j are equal to the average correlation of return, $\overline{\rho}$, on securities; and (iii) the investor allocates 1/N of his wealth to all securities.

11. Under these assumptions, the variance of return on the portfolio, V_p, can be written as:

$$V_p = \overline{V}(\frac{1}{N}) + \frac{N(N-1)}{N^2}\overline{V\rho}. \tag{5}$$

The effect of variations in $\overline{V}$, $\overline{\rho}$, and N on portfolio variance, V_p, can be determined by calculating the partial derivative of V_p, with respect to each of these variables:

$$\begin{aligned} \frac{\partial V_p}{\partial \overline{V}} &= \frac{1}{N} + \frac{N(N-1)}{N^2}\overline{\rho} > 0 \quad \text{if} \quad \overline{\rho} \geq 0 \\ \frac{\partial V_p}{\partial \overline{\rho}} &= \frac{N(N-1)}{N^2}\overline{V} > 0 \\ \frac{\partial V_p}{\partial N} &= -\overline{V}(\frac{1}{N^2}) + \overline{V\rho}(\frac{1}{N^2}) = \overline{V}(\overline{\rho}-1)(\frac{1}{N^2}) \leq 0. \end{aligned} \tag{6}$$

These equations indicate that the portfolio variance of return can be reduced in three ways: (i) increasing the number of securities in the portfolio; (ii) choosing securities having returns that are less correlated with returns on other securities; and (iii) if $\overline{\rho}$ is greater than or equal to zero, choosing securities with low variance or standard deviation of returns.

12. The formulas for portfolio mean and variance, given by Equation (2) and Equation (3), require estimates of the mean, E_i, and variance, V_i, of return on each security, as well as the covariance, C_{ij}, of returns on each pair of securities. If there are N securities under

consideration, Equation (2) and Equation (3) require N mean estimates, N variance estimates, and N (N – 1)/2 distinct covariance estimates, for a total of 2N + N x (N-1)/2 estimates. To illustrate, assume that an analyst is considering 200 securities for possible inclusion in a portfolio. Then the analyst must estimate 200 mean values, 200 variance values, and 19,900 covariance values to implement the Markowitz mean-variance model. Without simplification, it is unlikely that the analyst could estimate these inputs cost effectively.

13. One way to reduce the large number of estimates required to implement the Markowitz mean-variance model is to apply the model to asset classes rather than to individual securities. For example, if the universe of securities is divided into large U.S. stocks, small U.S. stocks, global stocks, emerging market stocks, corporate bonds, long-term U.S. government bonds and Treasury bills, the number of input estimates would be reduced from 20,300 to 35. Given the importance of the asset mix decision and the significant reductions in required estimates obtainable by considering asset categories rather than individual securities, it is not surprising that the Markowitz model is frequently applied to asset categories rather than to individual securities.

14. Another way to reduce the input requirements of the Markowitz mean-variance model is to make one or more simplifying assumptions about the covariance structure of security returns. For example, if one assumes that: (i) the return on an individual security i is related to the return on a market index via the equation:

$$R_i = \alpha_i + \beta_i R_m + e_i \quad \text{for } i = 1,2,\cdots,N \qquad (7)$$

where R_i is the return on security i, R_m is the return on the market index, and e_i is a random error term; (ii) $E[e_i(R_m - \overline{R_m})] = 0$; and (iii) $E(e_i \times e_j) = 0$; then the means, variances, and covariances of securities' returns are given by:

$$E_i = \alpha_i + \beta_i E_m \quad \textbf{(8)}$$

$$V_i = \beta_i^2 V_m + V_{ei}, \quad \textbf{(9)}$$

$$C_{ij} = \beta_i \beta_j V_m. \quad \textbf{(10)}$$

15. Substituting Equations (8), (9), and (10) into Equations (2) and (3), we obtain the following equations for the mean and variance of return on a portfolio of securities:

$$E_p = \sum_i X_i (\alpha_i + \beta_i E_m) \quad \textbf{(11)}$$

$$V_p = \sum_i X_i^2 \beta_i^2 V_m + \sum_i \sum_{j \neq i} X_i X_j \beta_i \beta_j V_m + \sum_i X_i^2 V_{ei}. \quad \textbf{(12)}$$

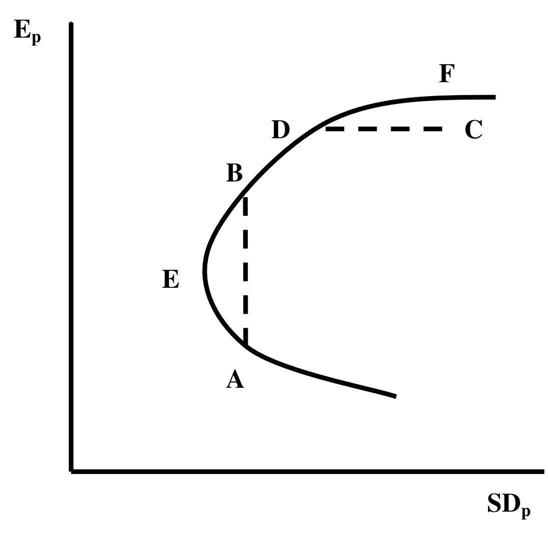
To estimate the mean and variance of return on any portfolio then, we need only to estimate the α_i, β_i, and V_{ei} inputs for each security and the expected return, E_m, and variance of return, V_m, on the market index. Thus, the total number of required estimates has been reduced from $2N + [N \times (N-1)]/2$ to 3N + 2. If the analyst is considering 200 securities for possible inclusion in a portfolio, the number of required estimates is reduced from 20,300 to 602.

B. The Feasible Set of Portfolios and the Efficient Frontier

16. The feasible set of portfolios is the set of all security allocations (X_1, … X_N) that satisfy the individual's portfolio constraints. An obvious portfolio constraint is that the sum of the proportion of wealth invested in all securities must equal 1. Other typical constraints are that the proportion invested in each security must be non-negative (that is, short selling is not allowed) and the investor will not invest more than a certain percentage of wealth in any one security.

17. The Markowitz mean-variance portfolio model allows an investor to translate all feasible portfolio proportions ($X_1, \ldots X_N$) into feasible combinations of: (i) expected return and variance of return; or (ii) expected return and standard deviation of return. Figure I shows one such feasible set of E_p, SD_p, combinations. Consider portfolio A shown in Figure I. Rational Markowitz mean-variance investors would not choose to invest in portfolio A because they could achieve a higher expected return by investing in portfolio B without increasing the portfolio standard deviation. Similarly, rational investors would not invest in portfolio C because they could achieve a lower portfolio standard deviation by investing in portfolio D without sacrificing mean return. The efficient set of portfolios consists of all portfolios with the highest mean return for any given level of standard deviation of return and the lowest standard deviation of return for any given level of mean return. The curved line EBDF is the efficient frontier for the feasible set of risky portfolios shown in Figure I.

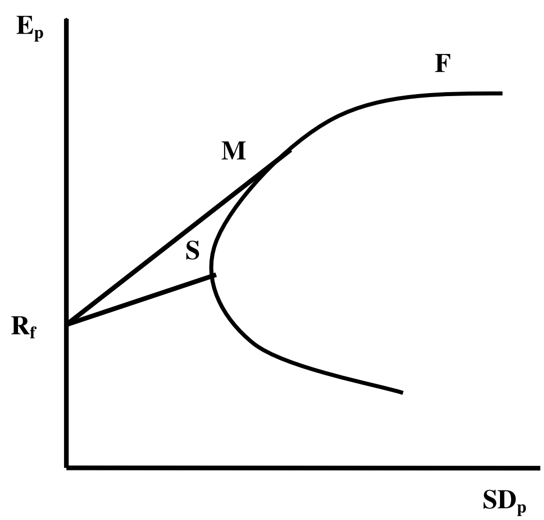
Figure I



C. The Effect of a Risk-free Security on the Shape of the Efficient Frontier

18. When all securities are risky, the efficient frontier typically has a shape similar to that shown in Figure I. Suppose now there exists a security with a risk-free rate of return, R_f. Since the risk-free security has zero variance, its mean-standard deviation combination lies on the vertical axis in mean-standard deviation space (see Figure II). In addition, the risk-free security must lie on the efficient frontier because there are no other securities with the same mean return and a lower standard deviation of return.

Figure II



19. To see how the shapes of the feasible set of securities and the efficient frontier change as a result of the presence of a risk-free security, consider a new portfolio consisting of the fraction X invested in a portfolio S of risky securities and a fraction (1-X) invested in the risk-free security. Because the return on the risk-free security has zero variance and is uncorrelated with the return on portfolio S, both the mean and standard deviation of return on the new portfolio are linearly related to the mean and standard deviation of return on portfolio S.[4]

[4] Specifically, $E_p = X \cdot E_s + (1 - X) R_f$ and $SD_p = X \cdot SD_s$.

Thus, the new portfolio lies somewhere on the straight line connecting R_f and S in Figure II, with its exact location depending on the fraction of wealth X invested in portfolio S.

20. Since the risky portfolio S in the above example is selected arbitrarily, the revised set of feasible portfolios consists of E_p, SD_p combinations lying on any line connecting R_f with a point such as S in the feasible set of risky portfolios. The slope of such a line is $(E_s – R_f)/SD_s$. Consider now the line connecting R_f with portfolio M in Figure II. The points on this line represent feasible portfolios with the fraction X invested in the risky portfolio M and the fraction (1-X) invested in the risk-free security. Since the slope $(E_M – R_f)/SD_M$ is greater than the slope $(E_s – R_f)/SD_s$ for any risky portfolio S, for any feasible portfolio not on the line connecting R_f and risky portfolio M, there exists a portfolio on the line connecting R_f and risky portfolio M with a higher mean return and the same standard deviation of return or the same mean return and a lower standard deviation of return. Thus, the points on the line connecting R_f and M are not only feasible but also efficient. Evidently, the new efficient frontier consists of the union of all points on the line connecting R_f and M and all points on the efficient frontier of risky securities between M and F.[5]

D. Identifying the Efficient Frontier

21. As noted above, the efficient set of portfolios consists of all portfolios with the highest mean return for a given level of variance or standard deviation and the lowest variance (standard deviation) for a given level of mean. Once the E_i, V_i, and C_{ij} inputs have been estimated, the analyst can calculate the mean-variance efficient frontier by solving the following optimization problem for all non-negative values of the parameter λ:

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[5] This conclusion strictly applies only when the investor cannot finance risky investments with borrowing. If the investor can borrow as well as lend at the risk-free rate, the efficient frontier will consist of the entire straight line emanating from R_f and extending through the tangency point M on the efficient frontier of risky securities.

$$\begin{aligned} &\text{Minimize } V_p - \lambda E_p \\ &\text{with respect to } X_1, X_2 \cdots, X_N \\ &\text{subject to } \sum_{i} X_i = 1 \\ &X_i \geq 0; i = 1, \cdots, N. \end{aligned} \tag{13}$$

22. The above problem is called the standard mean-variance portfolio selection problem. Non-standard forms of the mean-variance portfolio selection problem include cases where: (i) additional linear constraints apply, and (ii) the amounts invested in one or more securities may be negative. Markowitz (1956, 1959, and 2000) describes efficient algorithms for solving both standard and non-standard versions of the mean-variance portfolio selection problem. These algorithms are used extensively in practical mean-variance portfolio analysis.

23. Elton, Gruber, and Padberg (1976, 1978) demonstrate that an alternative simple procedure can be used to select mean-variance efficient portfolios when the single-index model, Equation (7), is accepted as the best method of forecasting mean-variance portfolio inputs. Their simple procedure requires that securities be ranked based on the ratio of their expected excess return to their beta:

$$(E_i - R_f) / \beta_i \tag{14}$$

where E_i is the expected return on security i, R_f is the return on the risk-free security, and β_i is the sensitivity of the return on security i to changes in the market index as measured by Equation (7). Elton, Gruber, and Padberg prove that all risky securities with an excess return to β ratio above a specific cut-off, C^*, should be included in a mean-variance efficient portfolio; and all risky securities with an excess return to β ratio below this cut-off value should be excluded. Formulas for C* for both the case where short sales are not permitted and the case where short sales are permitted are given in Elton, Gruber, and Padberg's papers.

E. Choosing the "Best" Portfolio on the Efficient Frontier

24. In the Markowitz mean-variance model, an investor should always choose a portfolio on the mean-variance efficient frontier. However, the investor's choice of the best portfolio on the efficient frontier depends on his or her attitude toward risk. Risk-averse investors will likely choose efficient portfolios near the minimum risk portfolio, E, on the efficient frontier in Figure I (or R_f on the efficient frontier in Figure II), while risk-tolerant investors will likely choose portfolios near the maximum mean portfolio, F.

25. But how does the investor actually make the choice of the "best" portfolio on the efficient frontier? It appears that there are two alternatives: direct choice, and investor utility functions. Direct choice involves a direct comparison of the mean and standard deviation of various portfolios on the efficient frontier. In the direct choice approach, the investor is simply presented with information on the means and standard deviations of various portfolios on the efficient frontier and asked to choose a preferred combination of mean and variance.

26. In contrast, investor utility functions involve an attempt to capture the investor's aversion to risk in the form of an investor-specific utility function:

$$U = E_p - kV_p, \tag{15}$$

where k is a parameter indicating the investor's aversion to risk, as measured by variance of return. In this approach, a portfolio advisor would estimate an investor's risk aversion parameter, k, from the investor's responses to a series of questions regarding the investor's attitude toward risk. Investors with high risk aversion would be assigned high values of k, while investors with low risk aversion would be assigned low values of k. The advisor would then calculate the utility, U, of each portfolio on the efficient frontier and recommend the portfolio with the highest utility.

27. In the tradition of Bernoulli (1738) and von Neuman and Morgenstern (1946), economists generally assume that investors wish to maximize their expected utility of wealth, an assumption that has allowed economists to derive a rich set of conclusions about investment behavior. However, practitioners have found that for most investors, utility functions are an impractical device for selecting portfolios. In their experience, they find that investors do not understand the concept of utility and are generally unable to provide the information required to determine their utility function analytically. While this conclusion probably explains the minimal use of utility functions in practical portfolio analysis, it does not rule out using utility functions to obtain practical insights into optimal investment policies for typical investors. Indeed, we demonstrate below how utility analysis has produced many useful guidelines for lifetime consumption-investment decision making.

F. Comments on the Mean-variance Model

28. The Markowitz mean-variance portfolio model has undoubtedly been one of the most influential models in the history of finance. Since its introduction in 1952, the mean-variance model has provided an intellectual foundation for much later research in finance. Because of its rich practical insights, the Markowitz model also continues to strongly influence practical financial management. Nonetheless, the mean-variance approach to portfolio selection is sometimes criticized because it implicitly assumes that information on a portfolio's mean return and variance of return is sufficient for investors to make rational portfolio decisions. Tobin (1958) notes that information on a portfolio's mean return and variance of return is only sufficient for rational portfolio decision making if: (i) the investor's utility function is quadratic; or (ii) the probability distribution of security returns is normal. Neither of these assumptions is likely to be strictly true.

29. Economists generally agree that a reasonable utility function should display non-satiety, risk aversion, decreasing absolute risk aversion, and constant relative risk aversion.[6] The problem with a quadratic utility function is that it displays satiety (that is, an investor with this utility function eventually prefers less wealth rather than more); and increasing absolute and relative risk aversion. Mossin (1968) demonstrates that the only utility functions that satisfy all four of the above desirable characteristics of utility functions are the logarithmic function, log W,[7] and the power function, $W^{1-\gamma}$. Thus, the assumption that it is rational for investors to evaluate risky choices based solely on the mean and variance of return cannot be strictly justified on the grounds that investors' utility functions are quadratic.

30. The other way to justify the assumption that investors base their risky choices solely on the mean and variance of returns is that security returns are normally distributed. But this justification is also problematic. The normal distribution is symmetric with a positive probability that returns can take any value on the real line. However, with limited liability, an investor can never lose more than his entire wealth—that is, $(1+r_t)$ must be greater than or equal to zero. In addition, the investor's multi-period return is the product of individual period returns, and the product of normally distributed variables is not normally distributed. Thus the rationality of relying solely on the mean and variance of portfolio returns cannot be strictly justified on the grounds that returns are normally distributed.

31. Markowitz (1959), Levy and Markowitz (1979), and Samuelson (1970), among other prominent economists, recognize that investor utility functions are unlikely to be quadratic and that security return distributions are unlikely to be normally distributed. However, they

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[6] The desirable attributes of utility functions are discussed more fully below.
[7] In this paper, we use the notation, log W, to indicate the natural logarithm of W.

defend the validity of the mean-variance approach to portfolio decision making based on the belief that one or more of the following statements is true:

- Within any interval, utility functions are approximately quadratic.
- Probability distributions can often be approximated by their first two moments.
- The mean-variance model, though not strictly rational, is nonetheless useful for investors because it provides information that investors find to be relevant and leads them to make better decisions than they would in the absence of the model.

Indeed, Markowitz and Levy and Markowitz demonstrate that many utility functions are approximately quadratic in any interval; Samuelson demonstrates that probability distributions can under fairly general conditions be approximated by their first two moments; and the prevalence of the mean-variance framework in practical decision making suggests that the mean-variance model is useful to investors.

III. The Geometric Mean Portfolio Model

32. As described above, Markowitz achieves simplicity in the mean-variance model by focusing on the economic trade-off between risk and return in a single-period world. However, many investors make portfolio decisions in a multi-period world where portfolios can be rebalanced periodically. For these investors, Latané (1959) recommends an alternative framework, the geometric mean portfolio model. He argues that the maximum geometric mean strategy will almost surely lead to greater wealth in the long run than any significantly different portfolio strategy, a result that follows from similar conclusions of Kelly (1957) in the context of information theory. Breiman (1960, 1961) states the precise conditions for which this result holds and develops additional properties of the geometric mean strategy. A recent news article

describes how two well-known fund managers, Edward Thorp and Bill Gross, have used the geometric mean portfolio strategy to improve the performance of their funds.[8]

A. The Geometric Mean Strategy and Long-run Wealth Maximization

33. Consider an investor who invests an amount, W_0, at the beginning of period 1 and earns a known rate of return on investment of $R_t = (1+r_t)$ in periods t = 1, …, T. If the investor reinvests all proceeds from his investment in each period, his wealth at the end of period T will be:

$$\begin{aligned} W_T &= W_0(1+r_1)(1+r_2)\cdots(1+r_T) \\ &= W_0\prod_t R_t. \end{aligned} \tag{16}$$

Let G=(1+g) denote the investor's compound average, or geometric mean, return on his investment over the period from 1 to T. Then,

$$\begin{aligned} G &= [(1+r_1)(1+r_2)\cdots(1+r_T)]^{1/T} \\ &= \prod_t R_t^{1/T}. \end{aligned} \tag{17}$$

Since $W_T = W_0G^T$, the investor's terminal wealth, W_T, will be maximized when the geometric mean return on investment, G, is maximized.

34. In practice, the returns $R_t = (1+r_t)$ for t = 1, …, T are uncertain at the time the investor makes the initial investment decision. Assume that the return on investment is independently and identically distributed, and that there are J possible outcomes for R. Let P_j denote the probability of obtaining the j^{th} return outcome. Then, the forward-looking geometric mean return on investment in this uncertain case is defined as:

[8] "Old Pros Size Up the Game, Thorp and Pimco's Gross Open Up on Dangers of Over-betting, How to Play the Bond Market," *The Wall Street Journal*, Saturday/Sunday, March 22-23, 2008.

$$G^E = \prod_j R_j^{P_j}. \tag{18}$$

In the geometric mean portfolio model, the investor's objective is to maximize the forward-looking geometric mean return on investment, G^E.

35. When analyzing a variable such as G or G^E that is equal to the product of other variables, it is frequently convenient to analyze the logarithm of the variable rather than the variable itself. From Equation (18), the log of G^E is equal to the expected log return on investment:

$$\begin{aligned} \log G^E &= \sum_j P_j \log R_j \\ &= E \log R. \end{aligned} \tag{19}$$

Because the log function is monotonically increasing throughout its domain, any portfolio that maximizes G^E will also maximize log G^E, and hence E log R. Thus, the geometric mean portfolio strategy is equivalent to a portfolio strategy that seeks to maximize the expected log return on investment, E log R.

36. Since the return on investment is assumed to be independently and identically distributed, the T values for R shown in the definition of G in Equation (17) can be considered to be a random sample of size T from the probability distribution for R. Let G^S denote the geometric mean return calculated from the random sample of size T from the probability distribution for R and log G^S denote the log of the sample geometric mean return. From Equation (17), log G^S is given by:

$$\log G^S = \frac{1}{T} \sum_t \log R_t. \tag{20}$$

According to the weak law of large numbers, the average of the sample values of a random variable will approach the expected value of the random variable as the sample size T approaches infinity. Presuming that the mean and variance of log R are finite, the weak law of large numbers assures that for any positive numbers ε and δ, no matter how small, there exists some positive number τ, perhaps large but nevertheless finite, such that for all T>τ,

$$\text{Prob}\left\{\left|\frac{1}{T}\sum_t \log R_t - E\log R\right| < \varepsilon\right\} \geq 1-\delta. \tag{21}$$

An alternate notation for this condition is

$$\text{plim}\left(\frac{1}{T}\sum_t \log R_t\right) = E\log R, \tag{22}$$

where plim denotes probability limit.

37. Let R^A denote the investor's returns under the maximum geometric mean strategy, and R^B denote the investor's returns under a significantly different strategy.[9] Then, from the above discussion, we know that:

$$\text{plim}\left(\frac{1}{T}\sum_t \log R_t^A\right) = E\log R^A, \tag{23}$$

$$\text{plim}\left(\frac{1}{T}\sum_t \log R_t^B\right) = E\log R^B, \tag{24}$$

and

$$E\log R^A > E\log R^B. \tag{25}$$

Thus,

$$\text{plim}\left(\frac{1}{T}\sum_t \log R_t^A\right) > \text{plim}\left(\frac{1}{T}\sum_t \log R_t^B\right). \tag{26}$$

[9] By "significantly different strategy," we mean a strategy that has a lower expected log return than the maximum geometric mean strategy.

This is turn implies that for T sufficiently large, it is virtually certain that $\left(\frac{1}{T}\right)\sum_t \log R_t^A$ will exceed $\left(\frac{1}{T}\right)\sum_t \log R_t^B$, and do so by an amount very nearly equal to (E log R^A – E log R^B) > 0. That is, in the long run, the investor will almost surely have greater wealth by using the geometric mean strategy than by using any significantly different strategy.

B. The Relationship between the Geometric Mean Strategy and the Log Utility Function

38. As shown above, the geometric mean portfolio strategy: (i) almost surely produces greater wealth in the long run than any significantly different strategy; and (ii) is equivalent to a strategy that seeks to maximize the expected log return. Further, Mossin (1968) demonstrates that for the log and power utility functions, maximizing the expected utility of return is equivalent to maximizing the expected utility of terminal wealth. Thus, for investors with log utility functions, the maximum geometric mean portfolio criterion is equivalent to the maximum expected utility of wealth criterion.

C. Desirable Properties of the Log Utility Function

39. For investors with log utility functions, the equivalence of the maximum geometric mean and the maximum expected log utility criteria is significant because log utility functions have many desirable properties. Among these properties are: (i) non-satiety; (ii) risk aversion; (iii) decreasing absolute risk aversion; (iv) constant relative risk aversion; (v) aversion to margin investing (that is, investing with borrowed money); and (vi) optimality of myopic decision making. Mossin (1968) demonstrates that the log and power functions are the only utility functions with all of these properties.

40. Non-satiety. Because the log utility function is increasing throughout its domain, investors with log utility functions always prefer more wealth to less, an attribute that economists refer to as "non-satiety." Although non-satiety would seem to be an obvious desirable property of a utility function, many utility functions do not possess this property, including the quadratic utility function.

41. Risk aversion. Consider the choice between (i) receiving \$W for certain and (ii) having a 50/50 chance of receiving either \$(W + C) or \$(W – C). Investors who choose alternative (i) over alternative (ii) are said to be risk averse, because these alternatives have the same expected value, but the second alternative has greater risk. Risk-averse investors choose alternative (i) because the difference in utility from receiving \$(W + C) rather than \$(W) is less than the difference in utility from receiving \$(W – C) rather than \$(W). That is, $U(W + C) - U(W) < U(W) - U(W - C)$. Evidently, risk-averse investors have utility functions characterized by $U''(W) < 0$.[10] Since the second derivative of log W is $-1/W^2$, investors with log utility functions are risk averse.

42. Decreasing absolute risk aversion. Although all risk-averse investors have utility functions characterized by $U''(W) < 0$, some investors are more risk averse than others. The intensity of an investor's aversion to risk is determined by the curvature of the investor's utility function, where curvature is measured by the ratio of $U''(W)$ to $U'(W)$. Specifically, Pratt (1964) and Arrow (1965) define the coefficient of absolute risk aversion by the equation:

$$ARA(W) = -\frac{U''(W)}{U'(W)}. \tag{27}$$

[10] We use the notation $U''(W)$ to indicate the second derivative of the utility function U with respect to its argument, W.

For small gambles, Pratt and Arrow demonstrate that ARA(W) determines the dollar amount an investor is willing to pay to avoid a fair gamble with the possibility of either winning or losing a constant absolute dollar amount, C. They argue that absolute risk aversion should decrease with wealth because rich investors would be relatively unconcerned with losing an amount, C, that would cause great concern for poor investors. For the log utility function, ARA(W) equals 1/W. Thus, log utility functions imply that absolute risk aversion declines as wealth increases.

43. Constant relative risk aversion. Pratt and Arrow define the coefficient of relative risk aversion by the equation:

$$RRA(W) = -\frac{WU''(W)}{U'(W)}. \qquad \textbf{(28)}$$

For small gambles, they demonstrate that RRA(W) determines the fraction of wealth that an investor will pay to avoid a fair gamble with a possibility of winning or losing a specific fraction, C/W, of wealth. Economists generally believe that relative risk aversion should remain constant as wealth increases. This belief is consistent with the evidence that interest rates and risk premia have remained constant as average wealth has increased over time. Since RRA(W) equals 1 for the log utility function, log utility functions display constant relative risk aversion.

44. Aversion to investing on margin. Investors can frequently enhance their expected return on investment by investing on margin. However, margin investing is considered to be risky for individual investors because it can greatly increase both the variability of return on investment and the probability of bankruptcy. Investors with log utility functions are averse to margin investing because margin investing increases the probability that wealth will be less than some small value ε greater than zero; and their utility of wealth approaches minus ∞ as W approaches zero.

45. Optimality of myopic decision making. As noted above, lifetime portfolio selection is generally a complex problem that, because of its dynamic interdependence, can only be solved through sophisticated dynamic programming procedures. However, Mossin (1968), Samuelson (1969), and Merton (1969) demonstrate that investors with either log or power utility functions can solve their lifetime portfolio selection problem one period at a time. For these investors, the decision that maximizes the expected utility of wealth at the end of period t is the same as the t^{th} period decision resulting from a dynamic optimization procedure that considers the effect of the individual's t^{th} period decision on all future decisions. Myopic decision making is a highly desirable property of utility functions because it allows analytical solutions to problems that would otherwise be impossible to solve.

D. Solving the Geometric Mean Portfolio Problem

46. An optimal geometric mean portfolio consists of a set of securities, i = 1, … N, and the optimal proportion of wealth, X_i, to invest in each security. Let $R_i = (1 + r_i)$ denote the return on security i. Then the optimal geometric mean portfolio can be found by solving the following optimization problem:

$$\begin{gathered} \text{Maximize } E\left[\log \sum_{i=1}^{N} R_i X_i\right] \\ \text{with respect to } X_1, \cdots, X_N \\ \text{subject to } X_i \geq 0, i = 1, \cdots, N \\ \sum_{i=1}^{N} X_i = 1. \end{gathered} \tag{29}$$

47. Vander Weide, Peterson, and Maier (1977) establish conditions required for the existence of a solution to the geometric mean portfolio problem and provide computational methods for finding exact solutions when solutions do exist. Maier, Peterson, and Vander Weide

(1977) examine the solutions to a relatively large number of simulated geometric mean portfolio problems obtained with the aid of a numerically efficient nonlinear programming code embedded within a partitioning algorithm. They find that the number of risky securities in an optimal geometric mean portfolio depends on one's expectations concerning future market conditions and on the conditions under which borrowing is permitted. When borrowing is not permitted, the investor who believes the market will fall should invest in just one security; the investor who believes the market will remain unchanged should diversify among two securities; and the investor who believes the market will rise should diversify among four to seven securities.[11]

48. When borrowing is allowed, the geometric mean portfolio problem must be modified to assure that the investor will not go bankrupt. Avoidance of bankruptcy can be accomplished by requiring the investor to withhold sufficient capital from investment to cover interest and principal payments on the borrowing. In the modified geometric mean portfolio problem, the investor who believes the market will rise should choose the same securities in the same relative proportions as when no borrowing is allowed. Furthermore, the individual characteristics of securities contained in optimal geometric mean portfolios also depend on one's assumptions about market conditions and the availability of borrowing. If a rising market is anticipated, the investor should invest in securities for which β_i and σ_i are large, and for which α_i is small. If the market is expected to decline, the investor should invest in stocks with high α_i and low β_i and σ_i.

49. Maier, Peterson, and Vander Weide (1977) also describe several heuristic portfolio building rules that provide near-optimal solutions to the geometric mean portfolio problem, including a geometric mean rule, a reward-to-variability rule, a reward-to-non-

[11] These numbers of securities are obtained under the assumption that the investor has 100 securities from which to choose. If the investor can choose among a greater number of securities, the optimal number to hold is likely to increase.

diversifiable variability rule, and a Kuhn-Tucker rule. Each rule follows the same principle: rank each security on the basis of one criterion, and then allocate equal dollar amounts to several of the top-ranked securities. They find that the geometric mean rule, the reward-to-variability rule, and the Kuhn-Tucker rule provide reasonable approximations to the returns on the optimal geometric mean portfolio.

50. Of course, the geometric mean portfolio problem cannot be solved without appropriate data inputs. Maier, Peterson, and Vander Weide (1977) note that the data inputs to the geometric mean portfolio problem can be estimated by assuming that the distribution of the holding period return, R_i, is related to a market index, I, through the equation:

$$\log R_i = \alpha_i + \beta_i I + \varepsilon_i, \qquad \textbf{(30)}$$

where I is defined as the expected value over all securities of the logarithm of the holding period return, α_i and β_i are constants, and ε_i is a normal random variable with mean zero and variance σ_i^2. The index I is considered a normal random variable whose parameters are chosen subjectively by the investor. Maier, Peterson, and Vander Weide (1982) develop an empirical Bayes estimation procedure for obtaining a simultaneous estimate of the three market model parameters of Equation (30) that makes use of more information than other estimates described in the literature.

E. Relationship between the Maximum Geometric Mean Return and the Mean and Variance of Return on a Portfolio

51. The geometric mean portfolio strategy is specifically designed for investors who wish to maximize their long-run wealth. Since the "long run" may be many years in the future, however, and mean-variance efficient portfolios have desirable short-run properties, it is natural to inquire whether the maximum geometric mean portfolio is mean-variance efficient.

Markowitz (1959) and Young and Trent (1969) address this inquiry by examining the expected value of several Taylor series approximations of either E log R or G. In discussing their methods and results, we will use the following notation:

μ_1	=	E(R)	=	the expected value or first moment of the probability distribution of R,
μ_2	=	$E(R- \mu_1)^2$	=	the variance or second central moment,
μ_3	=	$E(R- \mu_1)^3$	=	the skewness or third central moment, and
μ_4	=	$E(R- \mu_1)^4$	=	the kurtosis or fourth central moment.

52. The Taylor series approximation of log R centered around the mean value, μ_1, of R is given by:

$$\log R = \log \mu_1 + \frac{R-\mu_1}{\mu_1} - \frac{(R-\mu_1)^2}{2\mu_1^2} + \frac{(R-\mu_1)^3}{3\mu_1^3} - \frac{(R-\mu_1)^4}{4\mu_1^4} + \cdots \qquad \textbf{(31)}$$

Taking expectations of both sides of Equation (31), and noting that $E(R - \mu_1) = 0$, we then have:

$$\mathrm{E}\log R = \log \mu_1 - \frac{\mu_2}{2\mu_1^2} + \frac{\mu_3}{3\mu_1^3} - \frac{\mu_4}{4\mu_1^4} + \cdots \qquad \textbf{(32)}$$

53. Equation (32) provides several important insights about the relationship between the maximum geometric mean return and the mean and variance of return on a portfolio. First, if the third and higher moments of the probability distribution of R are “small” in relation to the first moment, E log R can be reasonably approximated by the expression:

$$\mathrm{E}\log R = \log \mu_1 - \frac{\mu_2}{2\mu_1^2}. \qquad \textbf{(33)}$$

Second, if Equation (33) is a reasonable approximation for E log R, the geometric mean portfolio will be approximately mean-variance efficient because E log R will be maximized when the mean, μ_1, is maximized for any value of variance, μ_2, and the variance, μ_2, is minimized for any value of mean, μ_1. Third, if the third and higher moments of the probability distribution for R are not "small" in relation to the first moment, the geometric mean portfolio may not be mean-variance efficient. An example where the geometric mean portfolio is not mean-variance efficient is provided by Hakansson (1971).

54. To test whether the maximum geometric mean portfolio is approximately mean variance efficient, Markowitz (1959) examines the ability of two geometric mean approximations to E log R to predict the actual geometric mean return on nine securities and two portfolios over the period 1937 to 1954. The two geometric mean approximations include:[12]

$$G(1) = \mu_1 - \frac{\mu_1^2 + \mu_2^2}{2}, \text{ and}$$

$$G(2) = \log \mu_1 - \frac{\mu_2}{2\mu_1^2}.$$

He finds that G(1) consistently underestimates the geometric mean return on the nine securities and two portfolios over the period, with an average error of eight percent. However, G(2) performs significantly better than G(1). It slightly overestimates the actual geometric mean return with an average error of only 1.7 percent. From his analysis of these approximations, Markowitz suggests that G(2) be used to estimate the geometric mean return for each portfolio on the mean-variance efficient frontier. He advises investors never to choose portfolios on the mean variance efficient frontier with greater single-period means than the optimal geometric mean portfolio because such portfolios will have higher short-run variance than the optimal geometric mean portfolio and less wealth in the long run.

55. Using methods similar to Markowitz, Young and Trent (1969) empirically test the ability of five geometric mean approximations to predict the actual geometric mean return on

[12] G(1) is derived from a Taylor series approximation centered on R = 1, while G(2) is the approximation shown in Equation (33).

233 securities and various portfolios based on these securities. The five geometric mean approximations include:[13]

$$G(1) = (\mu_1^2 - \mu_2)^{1/2},$$

$$G(2) = \mu_1 - \frac{\mu_2}{2\mu_1},$$

$$G(3) = \mu_1 - \frac{\mu_2}{2},$$

$$G(4) = \mu_1 - \frac{\mu_2}{2\mu_1} + \frac{\mu_3}{3\mu_1^2}, \text{ and}$$

$$G(5) = \mu_1 - \frac{\mu_2}{2\mu_1} + \frac{\mu_3}{3\mu_1^2} - \frac{\mu_4}{4\mu_1^3}.$$

Using monthly holding period returns for the time period January 1957 to December 1960 and annual holding period returns for the period January 1953 to December 1960, they demonstrate that geometric mean approximations such as G(2) and G(3), based only on the mean and variance of the probability distribution of R, provide predictions of geometric mean returns that differ on average from actual geometric mean returns by 0.5 percent. Thus, for their data set, we may conclude that maximum geometric mean portfolios are highly likely to be mean-variance efficient.

F. Comments on the Geometric Mean Portfolio Model

56. The geometric mean portfolio strategy is designed for investors who seek to maximize the expected value of their wealth in the long run. However, Merton and Samuelson (1974) demonstrate that maximizing the expected value of long-run wealth is not the same as maximizing the expected utility of long-run wealth. Since wealth at the end of a typical lifetime

[13] These approximations are derived from the Taylor series expansion of G centered around the mean value of R.

is variable, investors who are more risk averse than investors with log utility functions may prefer an alternative investment strategy that provides a stronger hedge against values of wealth that are less than the expected value of long-run wealth.

IV. Lifetime Consumption-Investment Model

57. The mean-variance and geometric mean portfolio models are designed to help investors choose the optimal proportions of wealth to invest in each security, based only on information regarding the probability distributions of returns on securities. If the probability distributions of returns are assumed to be independently and identically distributed, these models will recommend that the proportion of wealth invested in each security remain constant over the investor's lifetime. However, a constant proportion investment strategy is inconsistent with conventional wisdom that investors, as they age, should lower the proportion of wealth invested in risky stocks versus less risky bonds. The lifetime consumption-investment model is designed to help investors understand the conditions under which their optimal investment policy might change over their lifetimes, even if their probability beliefs remain constant.

58. Interest in lifetime consumption-investment models began in the late 1960s. Important early papers include Samuelson (1969), Merton (1969), Mossin (1968), and Hakansson (1969, 1970). Important later papers include Viceira (2001), Heaton and Lucas (2000), Koo (1998, 1999), Campbell, Cocco, Gomes, and Maenhout (2001), Bodie, Merton, and William Samuelson (1992), and Campbell and Cochrane (1999). Campbell and Viceira (2002) contains an excellent discussion of lifetime consumption-investment models, as well as a review of the literature on this important topic.

A. The Standard Lifetime Consumption-Investment Model

59. Consider an individual who must choose the amounts to consume, C_t, the fraction of wealth to invest in risky assets, w_t, and the fraction of wealth to invest in a risk-free asset, $(1 - w_t)$, at the beginning of each period (t = 0, 1, …, T). Assume that the individual's goal is to maximize the expected present value of the utility from lifetime consumption and that wealth must be either consumed or invested in each period. Let Z_t denote the random return on the risky asset, ρ the investor's discount rate, and r, the return on the risk-free asset. Then, the individual's standard lifetime consumption-investment problem can be stated as:[14]

$$\begin{gathered} \text{Max } E\left[\sum_{t=0}^{T}(1+\rho)^{-t}U(C_t)\right] \\ \text{with respect to } C_t, w_t \\ \text{subject to } \mathrm{C}_t = \left[W_t - \frac{W_{t+1}}{(1+r)(1-w_t)+w_t Z_t}\right] \\ W_0 \text{given, } \mathrm{W}_{\mathrm{T+1}} \text{ prescribed.} \end{gathered} \tag{34}$$

B. Analysis of the Optimal Lifetime Consumption-Investment Strategy

60. The standard formulation of the lifetime consumption-investment problem is difficult to solve without some simplifying assumptions. In his first paper on this subject, Samuelson (1969) assumes that (i) the individual's utility function displays constant relative risk aversion, that is, the utility function is either a log or power function; and (ii) the probability distribution for Z_t is independently and identically distributed. To his surprise, he finds that the optimal proportion to invest in the risky asset is constant under these assumptions. Thus, under the standard assumptions, the lifetime consumption-investment model produces the same

[14] This formulation is taken from Samuelson (1969).

constant proportion recommendation as the mean-variance and geometric mean models. Merton, Leland, Mossin, and Hakansson reach similar conclusions.

61. However, Samuelson (1989), Bodie, Merton, and William Samuelson (1992), and the authors of later papers cited above, demonstrate that when the standard assumptions of the lifetime consumption-investment model are modified to include non-tradable human wealth,[15] subsistence levels of consumption, and mean-reverting probability distributions of returns, the conclusion that the percentage invested in risky assets is constant must be modified. Since the literature on the effect of these additional variables on the optimal solution to the lifetime consumption-investment problem is complex, we limit our discussion here to a brief summary of relevant conclusions.

62. Non-tradable human wealth. The effect of human wealth on an individual's optimal investment strategy depends on whether human wealth is riskless or risky. Assume first that human wealth is riskless, that is, that the present value of an individual's future income is certain. Since riskless human wealth is equivalent to an implicit investment in a riskless asset, the investor should adjust the proportion of financial wealth, F_t, invested in risky assets to reflect the investor's implicit additional holding of riskless assets. When human wealth is riskless, Campbell and Viceira demonstrate that the optimal proportion of financial wealth, F_t, to invest in risky assets is an increasing function of the ratio of human wealth to financial wealth, H_t/F_t. This ratio will typically vary over an individual's lifetime.

63. For young investors, the ratio of human wealth to financial wealth will typically be high because the young investor (i) can expect to earn labor income for many years to come;

[15] Human wealth, H_t, reflects the expected present value of an individual's future income. Human wealth is non-tradable because the legal system forbids trading in claims on an individual's future income. Financial wealth, F_t, reflects the current market value of an individual's financial assets, that is, stocks and bonds. Total wealth, W_t, is equal to H_t plus F_t.

and (ii) has not had much time to accumulate financial wealth. Thus, young investors should allocate a relatively large percentage of financial wealth to risky assets. In contrast, for investors nearing retirement, the ratio of human wealth to financial wealth will typically be low. Thus, when human wealth is riskless, the percentage of financial wealth invested in risky assets should decline with age.

64. Assume now that labor income, and hence human wealth, is risky. If labor income is uncorrelated with the return on risky assets, the investor with human wealth should still invest a greater percentage of financial wealth in risky assets than the investor without human wealth. However, the percentage invested in risky assets should decrease with increases in the variance of labor income, that is, investors with high variance in labor income should reduce the percentage of financial wealth invested in risky assets to hedge some of the risk of their labor income.

65. If, on the other hand, labor income is perfectly positively correlated with the return on one or more risky financial assets, human wealth is an implicit investment in these financial assets. In this case, the investor should either increase the percentage of financial wealth invested in the riskless asset or diversify into risky assets that are uncorrelated with labor income. This latter conclusion applies to all individuals who hold a high percentage of financial wealth in the stock of their employer.

66. Human wealth also affects the optimal percentage of financial wealth to invest in risky assets through the investor's ability to vary his or her work effort. If the investor can increase work effort to offset losses on financial assets, the optimal percentage of financial wealth to invest in risky assets will increase.

67. Subsistence levels of consumption. The optimal percentage of financial wealth to invest in risky assets also depends on the investor's desire to maintain a minimum level of consumption. A minimum level of consumption may be thought of as negative income because a certain part of income must be set aside to assure the minimum level of consumption. Samuelson (1989) establishes that as the investor nears retirement, the investor should increase the allocation of financial wealth to risk-free bonds in order to assure a minimum level of consumption in retirement. The shift towards risk-free bonds arises as a result of the investor's need to provide a steady income stream in retirement to cover the minimum level of consumption. However, Merton (1969) notes that young investors may also have a need to assure a minimum level of consumption in the face of uncertain human and financial wealth. He establishes that this need would also shift the optimal portfolio of young investors toward riskless bonds. Constantinides (1990) and Campbell and Cochran (1999) analyze the case where the minimum level of consumption itself may depend on either the individual's prior consumption habits or the consumption norms of society.

68. Mean-reverting probability distribution of returns. Samuelson (1989) demonstrates that when asset returns are mean reverting, investors with long investment horizons should invest more in risky assets than investors with short investment horizons. Campbell, Cocco, Gomes, Maenhout, and Viceira (2001) show that if asset returns are mean reverting, investors should reduce the percentage of financial wealth invested in risky assets when the returns on risky assets have recently been above the long-run mean return and increase the percentage of financial wealth invested in risky assets when returns on risky assets have recently been below the long-run mean return. Thus, investors who believe that returns on risky assets

are mean reverting should also vary the percentage of financial wealth invested in risky assets with the status of the capital markets.

Bibliography

Arrow, K. J., 1965. Aspects of the theory of risk-bearing (Yrjö Jahnssonin Säätiö, Helsinki, Finland).

Bernoulli, Daniel, 1738, Exposition of a new theory on the measurement of risk, Papers of the Imperial Academy of Sciences in Petersburg V, 175-192.

Bodie, Zvi, Robert C. Merton, and William F. Samuelson, 1992, Labor supply flexibility and portfolio choice in a life cycle model, Journal of Economic Dynamics and Control 16, 427-449.

Breiman, Leo, 1960, Investment policies for expanding businesses optimal in a long-run sense, Naval Research Logistics Quarterly 7, 647-651.

Breiman, Leo, 1961, Optimal gambling systems for favorable games, in Jerzy Neyman, ed.: Proceedings of the Fourth Berkeley Symposium on Mathematics and Probability (University of California Press).

Campbell, John Y., João Cocco, Francisco Gomes, and Pascal Maenhout, 2001, Investing retirement wealth: a life-cycle model, in John Y. Campbell and Martin Feldstein, ed.: *Risk Aspects of Investment-based Social Security Reform* (University of Chicago Press, Chicago).

Campbell, John Y., João Cocco, Francisco Gomes, and Pascal Maenhout, 2001, Stock market mean reversion and the optimal equity allocation of a long-lived investor, *European Finance Review* 5.

Campbell, John Y., and Luis M. Viceira, 2002. *Strategic Asset Allocation: Portfolio Choice for Long-term Investors* (Oxford University Press, New York).

Campbell, John Y. and John H. Cochrane, 1999, By force of habit: a consumption-based explanation of aggregate stock market behavior, *Journal of Political Economy* 107, 205-251.

Chapman, S. J., 2006, The Kelly criterion for spread bets, IMA Journal of Applied Mathematics 72, 43-51.

Elton, Edwin J., and Martin J. Gruber, 1974, On the maximization of the geometric mean with lognormal return distribution, Management Science 21, 483-488.

Elton, Edwin J., and Martin J. Gruber, 1974, On the optimality of some multiperiod portfolio selection criteria, The Journal of Business 47, 231-243.

Elton, Edwin J., Martin J. Gruber, and Manfred W. Padberg, 1976, Simple criteria for optimal portfolio selection, The Journal of Finance 31, 1341-1357.

Elton, Edwin J., Martin J. Gruber, and Manfred W. Padberg, 1978, Simple criteria for optimal portfolio selection: tracing out the efficient frontier, The Journal of Finance 33, 296-302.

Fabozzi, Frank J., Francis Gupta, and Harry M. Markowitz, 2002, The legacy of modern portfolio theory, Journal of Investing 11, 7.

Fama, Eugene F., 1970, Multiperiod consumption-investment decisions, The American Economic Review 60, 163-174.

Grauer, Robert R., 1981, A comparison of growth optimal and mean variance investment policies, The Journal of Financial and Quantitative Analysis 16, 1-21.

Hakansson, Nils H., 1969, Optimal investment and consumption strategies under risk, an uncertain lifetime, and insurance, International Economic Review 10, 443-466.

Hakansson, Nils H., 1970, Optimal investment and consumption strategies under risk for a class of utility functions, Econometrica 38, 587-607.

Hakansson, Nils H., 1971, Capital growth and the mean-variance approach to portfolio selection, The Journal of Financial and Quantitative Analysis 6, 517-557.

Hakansson, Nils H., and Tien-Ching Liu, 1970, Optimal growth portfolios when yields are serially correlated, The Review of Economics and Statistics 52, 385-394.

Jacobs, Bruce I., Kenneth N. Levy, and Harry M. Markowitz, 2005, Portfolio optimization with factors, scenarios, and realistic short positions, Operations Research 53, 586-599.

Jacobs, Bruce I., Kenneth N. Levy, and Harry M. Markowitz, 2006, Trimability and fast optimization of long-short portfolios, Financial Analysts Journal 62, 36-46.

Jin, Hanqing, Harry Markowitz, and Zhou Xun Yu, 2006, A note on semivariance, Mathematical Finance 16, 53-61.

Kelly, J. L., 1956, A new interpretation of information rate, Bell Systems Technical Journal 35, 917-926.

Koo, Hyeng Keun, 1998, Consumption and portfolio selection with labor income: a continuous time approach, *Mathematical Finance* 8, 49-65.

Koo, Hyeng Keun, 1999, Consumption and portfolio selection with labor income: a discrete-time approach, *Mathematical Methods of Operations research* 50, 219-243.

Kroll, Yoram, Haim Levy, and Harry M. Markowitz, 1984, Mean-variance versus direct utility maximization, The Journal of Finance 39, 47-61.

Latané, Henry Allen, 1959, Criteria for choice among risky ventures, The Journal of Political Economy 67, 144-155.

Latané, Henry Allen, 1960, Individual Risk Preference in Portfolio Selection, The Journal of Finance 15, 45-52.

Latané, Henry A., 1963, Investment criteria: a three asset portfolio balance model, The Review of Economics and Statistics 45, 427-430.

Latané, Henry A., 1979, The geometric mean criterion continued, Journal of Banking & Finance 3, 309-311.

Latané, Henry A., and Donald L. Tuttle, 1966, Decision theory and financial management, The Journal of Finance 21, 228-244.

Latané, Henry A., and Donald L. Tuttle, 1967, Criteria for portfolio building, The Journal of Finance 22, 359-373.

Latané, Henry A., and William E. Young, 1969, Test of portfolio building rules, The Journal of Finance 24, 595-612.

Levy, H., and H. M. Markowitz, 1979, Approximating expected utility by a function of mean and variance, The American Economic Review 69, 308-317.

Maier, Steven F., David W. Peterson, and James H. Vander Weide, 1977, A Monte Carlo investigation of characteristics of optimal geometric mean portfolios, The Journal of Financial and Quantitative Analysis 12, 215-233.

Maier, Steven F., David W. Peterson, and James H. Vander Weide, 1982, An empirical Bayes estimate of market risk, Management Science 28, 728-737.

Markowitz, Harry M., 1952, Portfolio selection, The Journal of Finance 7, 77-91.

Markowitz, Harry M., 1959. Portfolio Selection: Efficient Diversification of Investments (Yale University Press, New Haven).

Markowitz, Harry M., 1976, Investment for the long run: new evidence for an old rule, The Journal of Finance XXXI, 1273-1286.

Markowitz, Harry M., 1987. Mean-variance analysis in portfolio choice and capital markets (B. Blackwell, New York).

Markowitz, Harry M., 1991. Portfolio selection : efficient diversification of investments (B. Blackwell, Cambridge, Mass.).
Markowitz, Harry M., 1999, The early history of portfolio theory: 1600 - 1960, Financial Analysts Journal 5-16.
Markowitz, Harry M., 1999, Normative portfolio analysis: past, present, and future, Journal of Economy and Business 42, 99-103.
Markowitz, Harry M., 2000. Mean-variance Analysis in Portfolio Choice and Capital Markets (J. W. Wiley).
Markowitz, Harry M., 2002, Efficient portfolios, sparse matrices, and entities: a retrospective, Operations Research 50, 154-160.
Markowitz, Harry M., 2005, Market efficiency: a theoretical distinction and so what?, Financial Analysts Journal 61, 17-30.
Markowitz, Harry M., and Erik L. van Dijk, 2003, Single-period mean-variance analysis in a changing world, Financial Analysts Journal 59, 30-44.
Markowitz, Harry M., and Andre F. Perold, 1981, Portfolio analysis with factors and scenarios, The Journal of Finance XXXVI, 871-877.
Merton, Robert C., 1969, Lifetime portfolio selection under uncertainty: the continuous-time case, *Review of Economics & Statistics* LI.
Merton, Robert C., and Paul A. Samuelson, 1974, Fallacy of the log-normal approximation to optimal portfolio decision-making over many periods, Journal of Financial Economics 1, 67-94.
Mossin, Jan, 1968, Optimal multiperiod portfolio policies, The Journal of Business 41, 215-229.
Pratt, J., 1964, Risk aversion in the small and in the large, Econometrica 32.
Roy, A. D., 1952, Safety first and the holding of assets, Econometrica 20, 431-499.
Samuelson, Paul A., 1969, Lifetime portfolio selection by dynamic stochastic programming, Review of Economics & Statistics 51, 239.
Samuelson, Paul A., 1970, The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments, The Review of Economic Studies 37, 537-542.
Samuelson, Paul A., 1979, Why we should not make mean log of wealth big though years to act are long, Journal of Banking & Finance 3, 305-307.
Samuelson, Paul A., 1989, A case at last for age-phased reduction in equity, Proceedings of the National Academy of Sciences of the United States of America 86, 9048-9051.
Sharpe, William F., 1963, A simplified model for portfolio analysis, Management Science 9, 277-293.
Sharpe, William F., 1967, Portfolio analysis, The Journal of Financial and Quantitative Analysis 2, 76-84.
Tobin, James, 1958, Liquidity preference as behavior towards risk, The Review of Economic Studies XXVI, 65-86.
Vander Weide, James H., David W. Peterson, and Steven F. Maier, 1977, A strategy which maximizes the geometric mean return on portfolio investments, Management Science 23, 1117-1123.
Vander Weide, James H., David W. Peterson, and Steven F. Maier, 1978, A comment on geometric mean portfolios: reply to Aucamp, Management Science 24, 860.
Viceira, Luis M., 2001, Optimal portfolio choice for long-horizon investors with nontradable labor income, *Journal of Finance* 56, 433-470.

von Neumann, John, and Oskar Morgenstern, 1946. Theory of Games and Economic Behavior (John Wiley & Sons, Inc., New York).

Young, William E., and Robert H. Trent, 1969, Geometric mean approximations of individual securities and portfolio performance, Journal of Financial & Quantitative Analysis 4, 179.

Ziemba, William T., 1972, Note on 'optimal growth portfolios when yields are serially correlated', Journal of Financial & Quantitative Analysis 7, 1995.